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Lifts of one-dimensional systems: I. Hyperbolic behaviour

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Abstract. We define the n-lift of a one-dimensional system $x_{i+1} = f(x_i)$. The n-lift can be thought of as a perturbation of the one-dimensional system depending on the state of the system $n-1$ time-steps back. We prove that certain f-invariant Cantor sets give invariant Cantor sets in the lifted system. We prove that if f has an invariant hyperbolic Cantor set then the lifted system has an invariant hyperbolic Cantor set provided the derivatives of f obey a simple condition. We also prove that hyperbolicity is preserved if the same conditions on the derivatives of f hold.

1. Introduction

This paper is the first of a three-part work on perturbations of one-dimensional discrete dynamical systems where the perturbation depends on several time-steps back. The first part is concerned with hyperbolic behaviour.

In this paper we will introduce the n-lift of a one-dimensional discrete dynamical system. In [J] we defined the lift of k-dimensional dynamical systems as k-parameter families of systems defined on a phase space of double dimension. The n-lift of a one-dimensional system will turn out to be a family of systems parameterized by two real parameters and is dependent on a bounded map g. The n-lift of a one-dimensional system is defined on a phase space of dimension n. The use of the name *lift* should not be confused with the lift of a map to the covering space of its target space. Below we will give a short physical motivation for studying perturbations of this kind.

Consider the one-dimensional system

$$x_{i+1} = f(x_i)$$

and a perturbation of this system by

$$x_{i+1} = f(x_i) + \alpha g(x_i, x_{i-1}, x_{i-2}, \ldots, x_{i-n+2}) + \epsilon x_{i-n+1}$$

where α and ϵ are small real parameters, and

$$g : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$$

is a smooth function. We assume that the function g has small C^1-size on bounded sets. Clearly this system is invertible if $\epsilon \neq 0$, and the smoothness properties depend only on the smoothness properties of f and g. The main purpose of this paper is to show that the dynamics of the one-dimensional system $x_{i+1} = f(x_i)$ and the dynamics of the perturbed system $x_{i+1} = f(x_i) + \alpha g(x_i, x_{i-1}, x_{i-2}, \ldots, x_{i-n+2}) + \epsilon x_{i-n+1}$ are closely related with respect to some properties, if g has small C^1-size and α and ϵ are small.

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Dynamical systems of the type

$$x_{i+1} = f(x_i) + \alpha g(x_i, x_{i-1}, x_{i-2}, \ldots, x_{i-n+2}) + \epsilon x_{i-n+1}$$

arise in many applications, in models in population dynamics, as models for Poincaré maps for attractors in forced oscillators and in data encoding in signal processing. However, most authors use well known maps such as the logistic map and the Hénon map in examples. Here we will outline a possible application for the systems studied in this paper.

Let $\{s_i\}$ be a sequence of signals. A sender wants to transmit coded signals to a receiver. This can be done by coding the signals through a chaotic attractor as

$$x_{i+1} = f(x_i) + \alpha g(x_i, x_{i-1}, x_{i-2}, \ldots, x_{i-n+2}) + \epsilon(s_{i-n+1})x_{i-n+1}.$$

The receiver can decode the received signal by applying the inverse map. Examples of this technique can be found in [E] where the logistic map and the Hénon map is used. During transmission the coded signals are subject to noise. Numerical experiments done by the author indicate that the term $g(x_i, x_{i-1}, x_{i-2}, \ldots, x_{i-n+2})$ may acts as a filter and reduce the error in the decoded signal.

Another important application of the systems studied in this paper is in reconstruction of attractors for systems where the nonlinearity is essentially one-dimensional. Hence it is of interest to study the mathematical properties of such systems.

Both applications will be discussed in a forthcoming paper.

Let us now introduce the variables $z_i^{(j)} = x_{i-j+1}$. Hence $z_{i+1}^{(j)} = x_{i-j+2}$, so the system

$$x_{i+1} = f(x_i) + \alpha g(x_i, x_{i-1}, x_{i-2}, \ldots, x_{i-n+2}) + \epsilon x_{i-n+1}$$

takes the form

$$\begin{aligned}
z_{i+1}^{(1)} &= f(z_i^{(1)}) + \alpha g(z_i^{(1)}, z_i^{(2)}, \ldots, z_i^{(n-1)}) + \epsilon z_i^{(n)} \\
z_{i+1}^{(2)} &= z_i^{(1)} \\
z_{i+1}^{(3)} &= z_i^{(2)} \\
&\vdots \\
z_{i+1}^{(n)} &= z_i^{(n-1)}.
\end{aligned}$$

We write this system as a mapping $\mathbb{R}^n \longrightarrow \mathbb{R}^n$, and hence it takes the form

$$(x_1, x_2, x_3, \ldots, x_n) \mapsto (f(x_1) + \alpha g(x_1, x_2, \ldots, x_{n-1}) + \epsilon x_n, x_1, x_2, \ldots, x_{n-1}).$$

We will denote this map by

$$F_{\alpha,\epsilon} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

or simply F. As remarked above, F is a diffeomorphism if f and g are C^1-maps and $\epsilon \neq 0$. The main result in this paper is to show a relationship between certain f-invariant sets on the real line, and certain F-invariant sets in $\mathbb{R}^n$. We will show that if C is an f-invariant Cantor set on the line such that $f|_C$ is topologically equivalent to a forward shift on c symbols, then, under suitable conditions F has an F-invariant Cantor set D such that the restriction $F|_D$ is topologically equivalent to a full shift on c symbols. We will also give a simple condition to ensure a hyperbolic structure on the tangent bundle over these sets.

The function g is used to allow a certain freedom in the perturbation, and the restriction in the form of the perturbation depending on x_n is used to have a contraction for ϵ small, and to obtain a nice formula for the inverse map.

2. The *n*-lift, the simple *n*-lift and the zero *n*-lift

We will now define some maps which are useful when studying perturbations of the system $f : \mathbb{R} \longrightarrow \mathbb{R}$ where the perturbation depends on several earlier states.

Definition. Let $f \in C^r(\mathbb{R}, \mathbb{R})$ and $g \in C^r(\mathbb{R}^{n-1}, \mathbb{R})$ where $r \geqslant 1$.
The zero n-lift of f is the map

$$(x_1, \dots, x_n) \mapsto (f(x_1), x_1, \dots, x_{n-1}).$$

The simple n-lift of f is the one-parameter family of maps given by

$$(x_1, \dots, x_n) \mapsto (f(x_1) + \epsilon x_n, x_1, \dots, x_{n-1}).$$

Any two-parameter family of maps of the form

$$(x_1, \dots, x_n) \mapsto (f(x_1) + \alpha g(x_1, \dots, x_{n-1}) + \epsilon x_n, x_1, \dots, x_{n-1})$$

is called a n-lift of f.

Remark. We note that the simple n-lift family contains the zero n-lift with $\epsilon = 0$, and that any n-lift family contains the simple n-lift family with $\alpha = 0$. The zero n-lift of f has the same dynamical properties as f. The simple form of the simple n-lift allows us to relate some of the dynamics of the zero n-lift to the dynamics of the simple n-lift (mostly by geometrical techniques). Once this is established we can use standard perturbation theorems for diffeomorphisms to establish the relation to a general n-lift of f.

3. Simple properties

We will now give some basic properties of the n-lifts of f. They are all easily proved by direct computation or by elementary theorems from calculus, so we omit the proofs here.

Proposition 1. Let $g \in C^1(\mathbb{R}^{n-1}, \mathbb{R})$ be fixed. The n-lift of f, denoted by F, has the following properties:

(1) If $f \in C^r(\mathbb{R}, \mathbb{R})$, $g \in C^r(\mathbb{R}^{n-1}, \mathbb{R})$ and $\epsilon \neq 0$ then $F \in \mathrm{Diff}^r(\mathbb{R}^n)$.
(2) The inverse of F, ($\epsilon \neq 0$), is given by

$$(x_1, x_2, \dots, x_{n-1}, x_n) \mapsto (x_2, x_3, \dots, x_n, \epsilon^{-1}(x_1 - f(x_2) - \alpha g(x_2, \dots, x_n))).$$

(3) F has a constant Jacobi determinant given by $\det DF = (-1)^{n+1}\epsilon$.
(4) The characteristic polynomial of the kth iterate of the zero n-lift of f at a point $p = (x_1, \dots, x_n)$ is given by

$$\det(\lambda I - DF^k(p)) = \left(\lambda - \frac{\partial f^k}{\partial x_1}(p)\right)\lambda^{n-1}.$$

(5) Assume f has a primitive p-periodic hyperbolic orbit $\{x_1, \dots, x_p\}$ and that the C^1-size of g is small. Then there exist numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that F has a p-periodic hyperbolic orbit for all $|\epsilon| < \delta_1$, $|\alpha| < \delta_2$. Furthermore if $\{x_1, \dots, x_p\}$ is stable then the corresponding lifted orbit is stable, and if $\{x_1, \dots, x_p\}$ is unstable, and $\epsilon \neq 0$ then the corresponding lifted orbit is a hyperbolic saddle with a one-dimensional unstable manifold and a $(n-1)$-dimensional stable manifold.

The zero-lift is given by the formula

$$(x_1, \ldots, x_n) \mapsto (f(x_1), x_1, \ldots, x_{n-1}).$$

Hence this map eventually forgets its past. We are concerned about p-periodic orbits. We have the following formulae for the p-power of the zero lift.

If $p < n$

$$(x_1, \ldots, x_n) \mapsto (f^p(x_1), f^{p-1}(x_1), \ldots, f(x_1), x_1, \ldots, x_{n-p})$$

while if $p \geqslant n$

$$(x_1, \ldots, x_n) \mapsto (f^p(x_1), f^{p-1}(x_1), \ldots, f^{p-n+1}(x_1)).$$

Let r be a p-periodic point for f and let q be a corresponding p-periodic point for F_0. Let $k_{p,n}$ be the unique integer such that $(k_{p,n} - 1)p < n \leqslant k_{p,n}p$. If r is a p-periodic point for f then it is clearly $k_{p,n}p$-periodic so we may replace p by $k_{p,n}p$ (and still denote this integer by p), and assume that the p-power of the zero-lift has the form

$$(x_1, \ldots, x_n) \mapsto (f^p(x_1), f^{p-1}(x_1), \ldots, f^{p-n+1}(x_1)).$$

The point $q \in \mathbb{R}^n$ is then given by

$$q = (r, f^{p-1}(r), f^{p-2}(r), \ldots, f^{p-n+1}(r))$$

where we use the convention of choosing r as the first coordinate in q. We see that any point in the affine-linear subspace of dimension $n-1$, given by $x_1 = r$ is mapped to q by F_0^p. Hence it acts as a (degenerate) stable manifold for q.

Assume that $|Df^p(r)| > 1$ and let $U_r \subset \mathbb{R}$ be a neighbourhood of r such that $|Df^p(x)| > 1$ for all $x \in U_r$. Then U_r is a local unstable manifold for r. Consider the map $\psi : \mathbb{R} \longrightarrow \mathbb{R}^n$ defined by

$$\psi(x) = (f^p(x), f^{p-1}(x), \ldots, f^{p-n+1}(x)).$$

We claim that $\psi(U_r)$ is a local unstable manifold for q. To see this we first note that $\psi(r) = q$, secondly that $F_0^p(\psi(U_r)) \supset \psi(U_r)$. The last statement is easily seen from the fact that image of $\mathbb{R}^n$ by F_0^n is contained in a one-dimensional set, $f^p(r) = r$ and f^p is expanding U_r. Let $\tilde{F}_0^p$ denote the restriction of F_0^p to $\psi(U_r)$. It is clear that $\tilde{F}_0^{-np}(s) \longrightarrow q$ as $n \to \infty$ for all $s \in \psi(U_r)$ since $f^{-np}(x) \longrightarrow r$ as $n \to \infty$ for all $x \in U_r$.

4. Homoclinic orbits

We now turn our attention to non-degeneracy homoclinic orbits associated with a hyperbolic periodic point of f. Let r be an unstable p-periodic orbit of f, and let U_r, ψ and q be as above. Recall that a homoclinic orbit is defined as an orbit starting in U_r hitting r after a finite number of iterates. The orbit is called non-degenerate if the derivative along this orbit is non-zero. We will show that non-degenerate homoclinic orbits in f give rise to transverse homoclinic points in the lift.

Let $r_h \in U_r$ and let $\{r_h, f(r_h), \ldots, f^h(r_h) = r\}$ be a non-degenerate homoclinic orbit for f associated with the periodic point p, i.e.

$$\prod_{i=0}^{h} Df(f^i(r_h)) \neq 0.$$

Consider an open neighbourhood $U_{r_h} \subset U_r$ of r_h. Then by the non-degeneracy condition above, we have $f^h(U_{r_h}) \supset W_r$, where $W_r \subset U_r$ is some local unstable manifold of r. Now let

$$q_h = (r_h, f^{p-1}(r_h), \ldots, f^{p-n+1}(r_h)).$$

Then $q_h \in \psi(U_r)$, and $F_0^h(q_h) = q$. To see this we simply note that

$$\begin{aligned} F_0^h(q_h) &= (f^h(f^p(r_h)), f^h(f^{p-1}(r_h)), \ldots, f^h(f^{p-n+1}(r_h))) \\ &= (f^p(f^h(r_h)), f^{p-1}(f^h(r_h)), \ldots, f^{p-n+1}(f^h(r_h))) \\ &= (r, f^{p-1}(r), f^{p-2}(r), \ldots, f^{p-n+1}(r)) = q. \end{aligned}$$

Now $q_h \in \psi(U_{r_h}) \subset \psi(U_r)$ and $F_0^h(\psi(U_{r_h}) \supset \psi(W_r)$. Let $W_{\mathrm{loc}}^s(F_0^p, q)$ denote the (degenerate) local stable manifold at q. This is a $(n-1)$-dimensional disc with centre in q and the first coordinate equal to r. Since $|Df^p(r)| \neq 0$, the tangent space of $\psi(W_r)$ has a non-zero first coordinate, and hence we have a transverse intersection of $F^h(\psi(U_{r_h}))$ and $W_{\mathrm{loc}}^s(F_0^p, q)$ at q.

Let $g \in C^1(\mathbb{R}^{n-1}, \mathbb{R})$ be fixed. Irwin's proof [PM] of the local stable manifold theorem produces a local stable manifold at a periodic point and depends only on forward iterates of the map. Hence it applies in the non-invertible case to produce a local stable manifold. The proof of the (un)stable manifold theorem using the nonlinear graph transform [Sh] produces a local unstable manifold at a periodic point, and depends only on forward iterates of the map. Hence it applies to the non-invertible case also. The maps F_0 and $F_{\alpha,\epsilon}$ are close in the C^1-topology on any compact set for any g with small C^1-size if the real numbers α and ϵ are close to zero. By invariant manifold theory [PM] we know that if D is an embedded disc in the unstable manifold and this disc contains the periodic point, then we may perturb the system such that the corresponding invariant manifold of the perturbed system contains a disc $\tilde{D}$ that is C^1-close to D. The zero-lift has a transverse intersection at q, and hence a small perturbation of the zero-lift has a transverse intersection close to q.

We have now proved the first part of the following theorem.

Theorem 1. Assume that $f : \mathbb{R} \longrightarrow \mathbb{R}$ has a non-degenerate homoclinic orbit associated with an unstable p-periodic orbit, and let $g \in C^1(\mathbb{R}^{n-1}, \mathbb{R})$ be fixed. Assume that the C^1-size of g is small. Then there exist numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that the n-lift of f has a transverse homoclinic point for all $|\alpha| < \delta_1$ and $0 < |\epsilon| < \delta_2$. In particular, a non-degenerate homoclinic orbit for f implies that the n-lift of f has an invariant hyperbolic set Λ containing the periodic orbit and the transverse homoclinic point such that the restriction of the lift to Λ is topologically equivalent to a sub-shift on finitely many symbols.

Proof. The first part is already proven. The second part can be proved simply by applying the Smale–Birkhoff theorem to our situation [GMN]. □

5. Invariant Cantor sets

We will now assume that $f : I \longrightarrow I$ has an invariant Cantor set C, such that the action of f on C is chaotic (in the sense described below), and such that C admits a construction of the following type.

Let $C_0 = [\zeta_0, \zeta_c]$ be the minimal closed interval containing C (here minimal is with respect to inclusion), where we assume f has the following properties.

We assume f has $c-1$, ($c \geqslant 2$), critical points, denoted by $\zeta_1, \ldots, \zeta_{c-1}$, in the interior, ordered by index, such that $\zeta_i < \zeta_{i+1}$. Furthermore we assume that f is monotone on each lap $L_i = [\zeta_{i-1}, \zeta_i]$, $i = 1, \ldots, c$, and that the critical values do not belong to C_0, i.e. $f(\zeta_i) \notin C_0$ for $i = 1, \ldots, c-1$, so $f(L_i) \supset C_0$.

We assume that the Cantor set C is the intersection over all its generations, denoted by C_n:

$$C = \bigcap_{n \geqslant 0} C_n$$

where

$$C_n = f^{-n}(C_0) \cap f^{-(n-1)}(C_0) \cap \cdots \cap f^{-1}(C_0) \cap C_0$$

where each C_n is a disjoint union of c^n closed intervals $I_{j_1 \cdots j_n}$

$$C_n = \bigcup I_{j_1 \cdots j_n}$$

where $j_k \in \{1, \ldots, c\}$, and where the indices are chosen such that $x \in I_{j_1 \cdots j_n}$ implies $f(x) \in I_{j_2 \cdots j_n}$. Furthermore, we assume that the Lebesgue measure, $\mu(C)$, of C is zero

$$\lim_{n\to\infty} \mu(C_n) = \lim_{n\to\infty} \mu\left(\bigcup I_{j_1 \cdots j_n}\right) = \lim_{n\to\infty} \sum \mu(I_{j_1 \cdots j_n}) = 0.$$

In particular each nested sequence of closed intervals converges to a unique point in C:

$$I_{j_1} \supset I_{j_1 j_2} \supset \cdots \supset I_{j_1 \cdots j_n} \supset \cdots \supset \{x_{\bar{j}}\}$$

where $\bar{j} = j_1 j_2 j_3 \cdots j_n \cdots$ is the unique address of $x_{\bar{j}} \in C$.

Let $h : C \longrightarrow \Sigma_c^+$ denote the address homeomorphism $x_{\bar{j}} \mapsto \bar{j}$, where Σ_c^+ is the space of all sequences on c symbols equipped with the metric

$$d(\bar{i}, \bar{j}) = \sum_{n=1}^{\infty} \frac{\delta(i_n, j_n)}{2^n}.$$

Hence our assumptions imply that $f : C \longrightarrow C$ is chaotic in the sense of being topologically conjugate to a forward shift on c symbols:

$$\begin{array}{ccc} C & \xrightarrow{f} & C \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ \Sigma_c^+ & \xrightarrow[\sigma]{} & \Sigma_c^+ \end{array}$$

where σ is the shift map on Σ_c^+.

We remark that we do not assume that f acts hyperbolically on C, i.e. $|Df(x)| > 1$ for all $x \in C$. However, we assume that $|Df(\zeta_i)| > 1$ for $i = 0, c$, to avoid the problem of one-sided stability of fixed points or period-two points at $\{\zeta_0, \zeta_c\}$.

Let us now turn to the construction of an invariant set for the simple n-lift assuming that f has an invariant set as described above. We will need some simple consequences of our assumptions. All of them can be trivially proved, so we omit the proofs, and collect the properties in a lemma.

Lemma 1. Assume that f, C and C_0 are as described in the text above.

(1) There exists a number $\delta_0 > 0$ such that

$$\min_{1 \leqslant i \leqslant c-1} d(f(\zeta_i), C_0) = \delta_0.$$

Here d denote the usual distance in $\mathbb{R}$.

(2) The boundary of C_0 is invariant, $f(\{\zeta_0, \zeta_c\}) \subset \{\zeta_0, \zeta_c\}$.

(3) There exist a positive number $\delta_1 > 0$ such that one of the following is true for all $0 < \delta < \delta_1$ (case (i) and (ii) can occur only if $c-1$ is odd, and case (iii) and (iv) can occur only if $c-1$ is even):
(i) $f(\zeta_0) = f(\zeta_c) = \zeta_0$ with $f(\zeta_0 - \delta) < \zeta_0 - \delta$ and $f(\zeta_c + \delta) < \zeta_0 - \delta$.
(ii) $f(\zeta_0) = f(\zeta_c) = \zeta_c$ with $f(\zeta_0 - \delta) > \zeta_c + \delta$ and $f(\zeta_c + \delta) > \zeta_c + \delta$.
(iii) $f(\zeta_0) = \zeta_0$ and $f(\zeta_c) = \zeta_c$ with $f(\zeta_0 - \delta) < \zeta_0 - \delta$ and $f(\zeta_c + \delta) > \zeta_c + \delta$.
(iv) $f(\zeta_0) = \zeta_c$ and $f(\zeta_c) = \zeta_0$ with $f(\zeta_0 - \delta) > \zeta_c + \delta$ and $f(\zeta_c + \delta) < \zeta_0 - \delta$.

Choose $\delta > 0$ such that $\delta < \min\{\delta_0, \delta_1\}$, where δ_0 and δ_1 are given in lemma 1. This δ will be kept fixed.

Let $C(\delta) = [\zeta_0 - \delta, \zeta_c + \delta]$. Assume, for simplicity, that f has a single critical point ζ_1 and that case (3) (i) in lemma 1 holds. Then there exists a $\epsilon_\delta > 0$ such that $f(\zeta_0 - \delta) + \epsilon t < \zeta_0 - \delta$, $f(\zeta_1) + \epsilon t > \zeta_c + \delta$ and $f(\zeta_c + \delta) + \epsilon t < \zeta_0 - \delta$ for all $t \in C(\delta)$ and all ϵ such that $|\epsilon| < \epsilon_\delta$. Clearly similar statements hold for the other cases in (3) in lemma 1 with any finite number of f-critical points. We will assume $0 < |\epsilon| < \epsilon_\delta$ in what follows.

Let

$$M_0(\delta) = \overbrace{C(\delta) \times \cdots \times C(\delta)}^{n \text{ times}}.$$

The set $M_0(\delta)$ is the n-dimensional analogue of $C(\delta)$ and will contain the F_ϵ-invariant set. The boundary of $M_0(\delta)$ consists of the faces of $M_0(\delta)$, denoted by $K_i^j(\delta)$, $i = 1, \dots, n$, $j = 0, 1$. Hence we write

$$\partial M_0(\delta) = \bigcup_{\substack{1 \leqslant i \leqslant n \\ j=0,1}} K_i^j(\delta)$$

where $x_i = \zeta_0 - \delta$ in $K_i^0(\delta)$ and $x_i = \zeta_c + \delta$ in $K_i^1(\delta)$. Clearly $K_i^j(\delta)$ is a $(n-1)$-dimensional cube. In what follows we will omit the explicit reference to δ in objects depending on δ.

Claim 1. $F_\epsilon(K_1^j) \cap M_0 = \varnothing$.

Claim 2. $F_\epsilon(M_0) \cap K_1^j = L_1^j \cup \cdots \cup L_c^j$ where $L_k^j \cap L_l^j = \varnothing$ if $k \neq j$. The dimension of L_k^j is $n-1$.

Claim 3.

$$F_\epsilon(M_0) \cap M_0 = \bigcup_{1 \leqslant i_0 \leqslant c} N_{i_0}$$

where $N_i \cap N_j = \varnothing$ if $i \neq j$. The dimension of N_i is n, and the sets L_i^j are parts of their boundaries.

Proof of claim 1. The number ϵ_δ is chosen such that the first component of $F_\epsilon(K_1^j)$ does not intersect $C(\delta)$. Hence the claim follows. □

Proof of claims 2 and 3. The intersections of the image of M_0 by K_1^j are determined by the equations $f(x_1) + \epsilon x_n = \zeta_0 - \delta$ and $f(x_1) + \epsilon x_n = \zeta_c + \delta$. There exist $2c$ intervals $Q_1, Q_2, \dots, Q_{2c} \subset C(\delta)$ such that if $x_1 \in Q_i$ then we can find a $x_n \in C(\delta)$ such that one of the above equations holds. Furthermore, by our choice of δ we see that $\zeta_0 - \delta, \zeta_1, \dots, \zeta_c + \delta \notin Q_i$ for $i = 1, \dots, 2c$. Hence claim 2 is proved. Let $J_i \subset C(\delta)$, $j = 1, \dots, c+1$ denote the intervals such that if $x_1 \in J_i$ then $f(x_1) + \epsilon x_n \notin C(\delta)$ for any $x_n \in C(\delta)$, and let $P_i \subset C(\delta)$, $i = 1, \dots, c$, denote the remaining parts of $C(\delta)$, i.e.

$$P_1 \cup \cdots \cup P_c = C(\delta) \setminus \left(\left(\bigcup Q_i\right) \cup \left(\bigcup J_k\right)\right).$$

Clearly the sets P_i corresponds to certain inner points in $F_\epsilon(M_0) \cap M_0$, and Q_i to cubes intersecting the boundary of M_0. □

Theorem 2. Assume $f : \mathbb{R} \longrightarrow \mathbb{R}$ has an invariant Cantor set, as described in the text above, on which f is conjugate to a forward shift on c symbols. Then the simple n-lift of f has an invariant set such that F_ϵ is conjugate to a full shift on c symbols provided ϵ is small enough and different from zero.

Proof. We define sets M_k and M_{-k} inductively by

$$M_k = F_\epsilon(M_{k-1}) \cap M_0 \quad \text{and} \quad M_{-k} = F_\epsilon^{-1}(M_{-k+1}) \cap M_0 \qquad \text{for} \quad k \geqslant 1.$$

Since F_ϵ is a diffeomorphism we see that

$$M_k = F_\epsilon^k(M_0) \cap F_\epsilon^{k-1}(M_0) \cap \ldots \cap F_\epsilon(M_0) \cap M_0$$

$$M_{-k} = F_\epsilon^{-k}(M_0) \cap F_\epsilon^{-k+1}(M_0) \cap \ldots \cap F_\epsilon^{-1}(M_0) \cap M_0$$

for $k \geqslant 1$. Hence $M_k \subset M_{k-1}$ and $M_{-k} \subset M_{-k+1}$.

Let $1 \leqslant l \leqslant k$. Then

$$\begin{aligned} F_\epsilon^l(M_{-k}) &= F_\epsilon^l\left(F_\epsilon^{-k}(M_0) \cap F_\epsilon^{-k+1}(M_0) \cap \ldots \cap F_\epsilon^{-1}(M_0) \cap M_0\right) \\ &= F_\epsilon^{-k+l}(M_0) \cap \ldots \cap F_\epsilon^{-1}(M_0) \cap M_0 \cap F_\epsilon(M_0) \cap \ldots \cap F_\epsilon^l(M_0) \\ &= M_{l-k} \cap M_l. \end{aligned}$$

Similarly, $F^{-l}(M_k) = M_{k-l} \cap M_{-l}$.

Let $\#M$ denote the number of connected components of a set $M \subset \mathbb{R}^n$. Note that if $M_k \neq \varnothing$ with $\#M_k = c^k$ then $M_{k-l} \cap M_{-l} \neq \varnothing$ with $\#(M_{k-l} \cap M_{-l}) = c^k$, and similar if $M_{-k} \neq \varnothing$ with $\#M_{-k} = c^k$ then $M_{l-k} \cap M_l \neq \varnothing$ with $\#(M_{l-k} \cap M_l) = c^k$, where the number of components is independent of l when $1 \leqslant l \leqslant k$. In particular, if $M_l \neq \varnothing$ with $\#M_l = c^{|l|}$ for all $l \in \mathbb{Z}$, then $M_k \cap M_{-k} \neq \varnothing$ with $\#(M_k \cap M_{-k}) = c^{2k}$ for all $k \geqslant 0$.

Since F_ϵ is a diffeomorphism then by construction M_l is closed for all $l \in \mathbb{Z}$. Assume in what follows that $M_l \neq \varnothing$ for all $l \in \mathbb{Z}$. Let $\Lambda_k = M_k \cap M_{-k}$ and let

$$\Lambda = \bigcap_{k \geqslant 0} \Lambda_k.$$

We see that

$$\Lambda_k = M_k \cap M_{-k} \subset M_{k-1} \cap M_{-k+1} = \Lambda_{k-1}$$

so Λ is a nested intersection of closed decreasing non-empty sets, and hence Λ is non-empty and closed. Clearly Λ is F_ϵ-invariant.

Let $p = (x_1, x_2, \ldots, x_n)$, and let A_i, $i = 1, 2, \ldots, c$, be the sets

$$A_i = \{p \in M_0 : \zeta_{i-1} < x_1 < \zeta_i\} \qquad \text{where} \quad i = 1, 2, \ldots, c.$$

We note that $M_1 = F_\epsilon(M_0) \cap M_0 \subset \cup A_i$. The *address* of a point $x \in M_k$ is the sequence $j_0 j_1 j_2 \cdots j_{k-1}$ of length k where $j_i = l$ if $F_\epsilon^i(x) \in L_l$. The address is well defined since $F_\epsilon^i(x) \in M_1$ for $i = 0, 1, \ldots, k-1$.

Now $F_\epsilon^l(M_{-k}) = M_{l-k} \cap M_l \subset M_1$ if $1 \leqslant l \leqslant k$. The *address* of a point $x \in M_k$ is the sequence $j_{-k} j_{-k+1} \cdots j_{-1}$ where $j_{-l} = i$ if $F^l(x) \in A_i$.

The *address* of $x \in M_{-k} \cap M_k$ is given by the sequence

$$j_{-k} j_{-k+1} \cdots j_{-1} j_0 j_1 j_2 \cdots j_{k-1}.$$

The addresses of two distinct points x, y belonging to the same connected component in $M_{-k} \cap M_k$ are clearly equal; however, if x, y belongs to different components they have different addresses. Hence each connected component of $M_{-k} \cap M_k$ is uniquely coded by a sequence of length $2k$ of c different symbols.

If every connected component of $M_{-k} \cap M_k$ converges to a point as $k \to \infty$, then we have the commutative diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{F_\epsilon} & \Lambda \\ h\downarrow & & \downarrow h \\ \Sigma_c & \xrightarrow{\sigma} & \Sigma_c \end{array}$$

where Σ_c denotes the space of bi-infinite sequences on c symbols equipped with its usual metric, σ is the shift-map and h is a homeomorphism.

We note that if $M_k \neq \varnothing$ with $\#M_k = c^k$ for $k \geqslant 1$ then $M_{-k} \neq \varnothing$ with $\#M_{-k} = c^k$ since $F_\epsilon^{-k}(M_k) = M_{-k}$. We claim that $M_k \neq \varnothing$ with $\#M_k = c^k$ for all $k \geqslant 0$, and will prove this by induction.

Assume M_{k-1} is a disjoint union of c^{k-1} n-dimensional cubes such that each cube in this union intersects K_1^j, $j = 0, 1$ in $(n-1)$-dimensional cubes. Let M denote one cube in this union. By the same argument as in the proofs of claims 1–3 above we see that $F_\epsilon(M) \cap M_0$ is a disjoint union of c n-dimensional cubes such that each cube intersects K_1^j, $j = 0, 1$, in $(n-1)$-dimensional cubes. Hence $M_k = F_\epsilon(M_{k-1}) \cap M_0$ has the desired properties, and since M_0 has these properties the claim follows by induction.

It remains to prove that every connected component in $M_k \cap M_{-k}$ converges to a point as $k \to \infty$.

We claim that each connected component of M_k converges to a curve with endpoints in K_1^0 and K_1^1 as $k \to \infty$. Let K_1^* denote the $(n-1)$-dimensional cube in M_0 defined by a constant first component, $x_1 = x^*$, where $\zeta_0 - \delta \leqslant x^* \leqslant \zeta + \delta$, and let M denote a connected component in M_k. We will prove that $M \cap K_1^*$ converges to a point as $k \to \infty$. This will indeed prove the claim.

Consider first a component in M_1 and its intersection with the (x_2, x_3)-plane in the cube K_1^* defined above. The indices refer to the coordinates in $\mathbb{R}^n$. The intersection is determined by the equation $x^* = f(x_1) + \epsilon x_n$, and there is a unique interval Q^* of x_1-values such that there is a $x_n \in C(\delta)$ for which this equation holds. Let q_1^* denote the length of Q^*, and let s denote the length of $C(\delta)$. By using the mean value of the derivative of f over the interval I_j in the one-dimensional system, where i_j is the interval corresponding to M^* we find that $q_1^* \approx \epsilon\mu(I_j)$. Hence $M \cap K_1^*$ is approximately a $(n-1)$-dimensional cube with the length of one side equal to $\epsilon\mu(I_j)$ and the length of the other sides equal to s. The total volume is therefor approximately $q_1^* c^{n-2}$. On the next iterate the side of length q_1 is mapped to x_3 and hence the second generation components are approximately cubes with two sides equal to q_1 and $n-3$ sides equal to c. After $n-1$ iterations the $(n-1)$th generation of components are approximately cubes with all sides equal to q_1. On the n-iterate we must modify the interval Q^*_{old} to Q^*_{new} because the intersection equation is now given by $x^* = f(x_1) + \epsilon x_n$ where $x_n \in Q^*_{\text{old}}$, and not in $C(\delta)$ as on earlier iterates, and we find that the length of Q^*_{new} is approximately $q_2^* = \epsilon\mu(I_{j_1 j_2})$ where $I_{j_1 j_2}$ is the corresponding component of the one-dimensional Cantor set. By continuing this process we clearly obtain cubes converging to a point since $\mu(I_{j_1 j_2 \cdots j_l}) \to 0$ as $l \to \infty$ by assumption. This proves the claim.

We will also need to control the size of the connected components in M_k. We claim that the connected components of M_{-k} converge to cubes of dimension $n-1$ transverse to the Cantor set of curves in $\cap M_k$. From the formula for the inverse map

$$(x_1, x_2, \ldots, x_n) \mapsto (x_2, x_3, \ldots, x_n, \epsilon^{-1}(x_1 - f(x_2)))$$

we see that the intersection equation can now be written as $x_n^* = \epsilon^{-1}(x_1 - f(x_2))$, i.e. $x_1 = f(x_2) + \epsilon x_n^*$. The set of useful x_2-values at a connected component in M_{-k} then has, clearly, length close to $\mu(I_{j_1 \dots j_k})$, where $I_{j_1 \dots j_k}$ is the interval corresponding to the component in question. As $k \to \infty$ the set of useful x_2-values converges to a Cantor set, and since these points are copied into the first component of the image, and all other components ranges all over $C(\delta)$ we have proved the claim. □

Corollary 1. Assume $f : \mathbb{R} \longrightarrow \mathbb{R}$ has an invariant Cantor set, as described in the text above, on which f is conjugate to a forward shift on c symbols. Then the n-lift of f has an invariant set such that F_ϵ is conjugate to a full shift on c symbols provided that the C^1-size of g, α and ϵ is small enough and ϵ is different from zero.

Proof. A modified version of the arguments in the proof of theorem 2 applies. □

6. Hyperbolic structures

Proposition 2. Let $F = F_{\epsilon_0}$ be a simple n-lift of f. Suppose Λ_{ϵ_0} is a compact F-invariant set. Let $p = (x_1, \dots, x_n) \in \Lambda$. If $|Df(x_1)| > \sqrt{2}$ for all $p \in \Lambda$ then the tangent bundle $T_{\Lambda_{\epsilon_0}}\mathbb{R}^n$ has a hyperbolic structure provided ϵ_0 is small enough.

Proof. Let $\xi \in T_p\mathbb{R}^n$ with usual coordinates $\xi = (\xi_1, \dots, \xi_n)$ such that the tangent map of the simple n-lift is given by

$$DF_\epsilon(p) = \begin{bmatrix} q & 0 & 0 & \cdots & 0 & \epsilon \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

where $q = Df(x_1)$. Then $DF_\epsilon(p)\xi = (q\xi_1 + \epsilon\xi_n, \xi_1, \dots, \xi_{n-1})$. Let $C_p \subset T_p\mathbb{R}^n$ be the cone

$$C_p = \left\{ \xi \in T_p\mathbb{R}^n \,:\, \xi_1^2 > \sum_{k=2}^{n} \xi_k^2 \right\}$$

and let τ denote the constant field of cones $p \mapsto C_p$. Since τ is constant we omit the base point p in C_p and write $C = C_p$. If $q^2 > 2$ then $q^2 - 2|q\epsilon| > 2$ for small ϵ. We will first prove that τ is DF_ϵ-invariant, i.e. $DF_\epsilon(p)C \subset C$. A simple computation show that

$$(q\xi_1 + \epsilon\xi_n)^2 \geqslant q^2\xi_1^2 - 2|q\epsilon||\xi_1||\xi_n| > (q^2 - 2|q\epsilon|)\xi_1^2 > 2\xi_1^2 > \sum_{k=1}^{n} \xi_k^2 \geqslant \sum_{k=1}^{n-1} \xi_k^2$$

and proves the DF_ϵ-invariance of τ. By the same argument we get that DF_ϵ is an expansion on C:

$$\|DF_\epsilon(p)\xi\|^2 = (q\xi_1 + \epsilon\xi_n)^2 + \sum_{k=1}^{n-1} \xi_k^2 > 2\xi_1^2 + \sum_{k=1}^{n-1} \xi_k^2 > \sum_{k=1}^{n} \xi_k^2 = \|\xi\|^2.$$

A simple computation shows that

$$DF_\epsilon^{-n+1}(p) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ \epsilon^{-1} & -\epsilon^{-1}q_1 & 0 & \cdots & 0 & 0 \\ 0 & \epsilon^{-1} & -\epsilon^{-1}q_2 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & \epsilon^{-1} & -\epsilon^{-1}q_{n-1} \end{bmatrix}$$

where the q_i, $i = 1, \dots, n-1$, are the derivatives of f evaluated along the orbit. Hence

$$DF_\epsilon^{-n+1}(p)\xi = (\xi_n, \epsilon^{-1}(\xi_1 - q_1\xi_2), \dots, \epsilon^{-1}(\xi_{n-1} - q_1\xi_n)).$$

We claim that $DF_\epsilon^{-n+1}(p)$ is an expansion on $T_p\mathbb{R}^n \setminus C$. To see this let us define singular slices P_i, $i = 1, \dots, n-1$, by

$$P_i = \{\xi \in T_p\mathbb{R}^n : -|\epsilon| \leqslant \xi_i - q_i\xi_{i+1} \leqslant |\epsilon|\}.$$

Let S^{n-1} denote the unit sphere in $T_p\mathbb{R}^n$. We define singular sets S_i, $i = 1, \dots, n-1$, on S^{n-1} by $S_i = S^{n-1} \cap P_i$. We claim that

$$S = \bigcap_{i=1}^{n-1} S_i \subset S^{n-1} \cap C.$$

To see this we consider the middle hyperplanes of the singular slices given by $\xi_{i+1} = q^{-1}\xi_i$, $i = 1, 2, \dots, n-1$, and note that $\xi_i = q_1^{-1}q_2^{-1} \cdots q_{i-1}^{-1}\xi_1$, $i = 2, \dots, n$, with

$$\sum_{i=1}^{n} \xi_i^2 = \left(1 + \sum_{k=1}^{n-1} \prod_{i=1}^{k} q_i^{-2}\right)\xi_1^2 = 1.$$

Hence

$$\xi_1^2 = 1 \Bigg/ \left(1 + \sum_{k=1}^{n-1} \prod_{i=1}^{k} q_i^{-2}\right).$$

Clearly $\xi \in S^{n-1} \cap C$ if and only if $\xi_1^2 > \frac{1}{2}$, and since $|q_i| > \sqrt{2}$ we find

$$\sum_{k=1}^{n-1} \prod_{i=1}^{k} q_i^{-2} < \sum_{k=1}^{n-1} \prod_{i=1}^{k} 2^{-1} = \sum_{k=1}^{n-1} 2^{-k} < 1.$$

Hence the intersection of the middle singular hyperplanes is contained in $S^{n-1} \cap C$, and by continuity

$$S = \bigcap_{i=1}^{n-1} S_i \subset S^{n-1} \cap C$$

if $|\epsilon|$ is small. This proves that $DF_\epsilon^{-n+1}(p)$ is an expansion on $T_p\mathbb{R}^n \setminus C$ since at least one of the terms $\epsilon^{-2}(\xi_i - q_i\xi_{i+1})^2 > 1$ in $\|DF_\epsilon^{-n+1}(p)\xi\|$ if $\xi \in S^{n-1} \setminus C$ by the argument above. Hence $\|DF_\epsilon^{-n+1}(p)\xi\| > \|\xi\|$. Hence τ satisfies theorem 2.2 of [GMN], and establishes a hyperbolic structure for the simple n-lift of f. □

The last theorem extends by the stability theorem for hyperbolic sets to n-lifts of f where the perturbation term g has small C^1-size.

Theorem 3. Let $F = F_{\epsilon_0}$ be a simple n-lift of f. Suppose Λ_{ϵ_0} is a compact F-invariant set. Let $p = (x_1, \dots, x_n)$, U an open neighbourhood of Λ_{ϵ_0} and $F_{\alpha,\epsilon}$ an n-lift of f such that the perturbation term g has small C^1-size. If $|Df(x_1)| > \sqrt{2}$ for all $p \in U$ then there exist numbers $\delta_i > 0$ $i = 1, 2$ such that $F_{\alpha,\epsilon}$ has a hyperbolic invariant set $\Lambda_{\alpha,\epsilon} \subset U$ for all ϵ with $|\epsilon - \epsilon_0| < \delta_1$ and $|\alpha| < \delta_2$. Furthermore, the dynamics on these sets are equivalent in the sense that the restriction to the invariant sets are all topologically conjugate.

Proof. By proposition 2 the $F_{0,\epsilon}$-invariant set Λ_{ϵ_0} has a hyperbolic structure. Hence the theorem follows from the stability theorem for hyperbolic sets [Nit]. □

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